Eur. Phys. J. B (2013) 86: 306
DOI: 10.1140/epjb/e2013-40331-9

THE EUROPEAN PHYSICAL JOURNAL B

Regular Article

Adaptive search for the optimal feedback gain of time-delayed feedback controlled systems in the presence of noise

Viktoras Pyragas[a] and Kestutis Pyragas

Center for Physical Sciences and Technology, A. Goštauto 11, 01108 Vilnius, Lithuania

Received 9 April 2013 / Received in final form 15 May 2013
Published online 3rd July 2013 –

Abstract. We propose two adaptive algorithms for the time-delayed feedback control method to tune the feedback gain to an optimal value in the presence of noise. By the optimal value we mean the value of the feedback gain that minimizes the mean square of the control signal. The first algorithm is model independent; it uses trial values of the feedback gain and defines the optimal value by the least-squares polynomial fitting. The second algorithm is based on the gradient descent method and requires the knowledge of the system equations. Here any initial value of the feedback gain is continuously adjusted towards the optimal value without any trials. The efficacy of the algorithms is demonstrated with different specific models, namely, a simple linear map, the Rössler system and the normal form of the subcritical Hopf bifurcation.

1 Introduction

The time-delayed feedback control (TDFC) algorithm has been introduced two decades ago [1] as a simple, robust and efficient tool for stabilization of unstable periodic orbits (UPOs) in nonlinear dynamical systems. The research devoted to the TDFC still remains one of the most active fields in applied nonlinear science [2]. The control signal in the TDFC algorithm is formed from a difference between the current state of the system and the state of the system delayed by one period of a target orbit. The method is asymptotically non-invasive because the control force vanishes whenever the target UPO is reached. The TDFC algorithm has been successfully implemented in quite diverse experimental systems from different fields of science. Some details of experimental implementations and various modifications of the TDFC algorithm up to 2006 are presented in the review paper [3]. In addition, we refer to recent theoretical and experimental results concerning the refuting of the odd number limitation of the TDFC algorithm [4–8], the global properties of the TDFC (the basins of attraction of the stabilized states) [6,9–11], the TDFC based bifurcation analysis for experiments [12], the TDFC with variable and distributed delays [13] and the phase-reduction-theory-based treatment of the TDFC algorithm [14,15]. An important practical application of the TDFC algorithm for an atomic force microscope has been recently demonstrated by Yamasue et al. [16].

One of the relevant issues in the application of the TDFC method is the adaptive search for the delay time, which should be equal to the period of actual UPO. This problem was solved by using e.g. the gradient descent method (see Ref. [17] and references therein). Here we consider another problem: the search for the optimal value of the feedback gain in the presence of noise. We introduce two techniques for solving this problem: a model-independent algorithm with the use of trial values of the feedback gain and the gradient descent algorithm, which does not use any trial points but requires the knowledge of the system equations.

For noiseless systems, the optimal feedback gain can be defined as that, which minimizes the leading Lyapunov exponent of the stabilized orbit. In the presence of noise, the system does not settle exactly on the stabilized orbit, but fluctuates in its vicinity, so that the criterion based on the Luapunov exponent becomes unapplicable. In this case the optimal feedback gain can be defined as that, which minimizes the mean squared difference between the current and delayed states of the system. We note that the latter definition is useless in the absence of noise, since in the noise-free system the above difference vanishes in the whole stability interval of the feedback gain. Thus the noise plays the crucial role in our analysis.

Note that some theoretical aspects of influence of noise on time-delayed feedback controlled discrete maps have been considered in references [18–20]. We also mention reference [21], where a speed-gradient method has been proposed for adaptive tuning of the feedback gain. The algorithm may shift the feedback gain from an unstable region into the stable one, whereas the convergence to the optimal value is not attained. Another adaptive algorithm that does not take into account the optimal value and the influence of noise has been considered in reference [22].

The rest of the paper is organized as follows. In Section 2 we describe a model independent algorithm, which uses trial values of the feedback gain and a least-squares

[a] e-mail: viktpy@pfi.lt

polynomial fitting (LSPF) in order to obtain online the optimal feedback gain of time-delayed feedback controlled systems in the presence of noise. We demonstrate this algorithm for a simple linear map as well as for the chaotic Rösssler system [23]. Section 3 is devoted to the gradient descent algorithm, which requires the knowledge of the model equations. This algorithm provides the convergence of the feedback gain to the optimal value without any trials. We present the formulation of the algorithm in a form of differential equations, which incorporate the feedback gain as a state dependent variable. The efficacy of the algorithm is demonstrated for the Rössler system and the normal form of the subcritical Hopf bifurcation. The paper is finished with the conclusions presented in Section 4.

2 Model independent algorithm

In this section, we describe a model independent algorithm for finding the optimal value of the feedback gain of the TDFC algorithm in the presence of noise.

2.1 Proportional feedback control of a noisy fixed point

To begin we consider an influence of noise on a simple problem of proportional feedback control of an unstable fixed point described by the following Langevin equation:

$$\dot{x}(t) = \lambda x(t) - kx(t) + \xi(t). \tag{1}$$

Here $\lambda > 0$ is the eigenvalue (or Lyapunov exponent) of the unstable fixed point $x = 0$, k is a feedback gain of the proportional control, and $\xi(t)$ is the white noise satisfying

$$\langle \xi(t) \rangle = 0, \quad \langle \xi(t)\xi(t') \rangle = \varepsilon^2 \delta(t - t'), \tag{2}$$

where $\delta(\cdot)$ is the Dirac delta-function and ε defines the strength of the noise. Solving equation (1) with the conditions (2) we obtain the following expression for the variance:

$$\langle x^2(t) \rangle = \langle x_0^2 \rangle e^{2\Lambda t} - \frac{\varepsilon^2}{2\Lambda} + \frac{\varepsilon^2}{2\Lambda} e^{2\Lambda t}. \tag{3}$$

Here $x_0 = x(0)$ is the initial condition and $\Lambda = \lambda - k$ is the Lyapunov exponent of the controlled fixed point. For $\Lambda > 0$ the variance grows to infinity while for $\Lambda < 0$ it saturates to a stationary value $\sigma^2 \equiv \langle x^2(t) \rangle_{t\to\infty}$, which is inversely proportional to the Lyapunov exponent Λ:

$$\sigma^2(k) = -\frac{\varepsilon^2}{2\Lambda} = \frac{\varepsilon^2}{2(k - \lambda)}. \tag{4}$$

Our aim is to find an optimal value of the feedback gain k, which provides the stabilization of the fixed point with the minimal variance σ^2. Here the variance is inversely proportional to the feedback gain and its minimum is attained at $k \to \infty$. Below we will see that in TDFC problems the optimal feedback gain has a finite value.

Equation (4) can be rewritten in the form

$$\Lambda(k) = -\frac{\varepsilon^2}{2\sigma^2(k)}, \tag{5}$$

which means that the Lyapunov exponent of the controlled noisy system can be obtained by measuring the variance $\sigma^2(k)$. Note that the dependence $\sigma^2(k)$ has a singular point at the boundary of the stability $k = \lambda$, while the Lyapunov exponent $\Lambda(k)$, which is inversely proportional to $\sigma^2(k)$, has a simple linear dependence on k. A simple dependence of the inverse function $1/\sigma^2(k)$ on the feedback gain k will be exploited below for an adaptive search of an optimal value of the feedback gain in more complex problems of time-delayed feedback control.

2.2 Time-delayed feedback control of a noisy linear map

Consider an influence of noise in the simplest TDFC problem, namely in the problem of stabilization of an unstable fixed point in a linear map:

$$x_{n+1} = \lambda x_n + k(x_n - x_{n-1}) + \xi_n. \tag{6}$$

Here λ is the eigenvalue of the fixed point $x = 0$, k is the feedback gain of the TDFC, and ξ_n is a white noise that satisfies:

$$\langle \xi_n \rangle = 0, \quad \langle \xi_n \xi_m \rangle = \varepsilon^2 \delta_{nm}, \tag{7}$$

where δ_{nm} is the Kronecker delta. The fixed point of the uncontrolled system is unstable if $|\lambda| > 1$. We consider the case of an unstable fixed point with $\lambda < -1$. The case $\lambda > 1$ cannot be stabilized with the TDFC due to the odd number limitation [24].

Denoting $y_n = x_{n-1}$, we can rewrite the map as

$$x_{n+1} = \lambda x_n + k(x_n - y_n) + \xi_n, \tag{8a}$$
$$y_{n+1} = x_n. \tag{8b}$$

First, we investigate this system analytically and then formulate an adaptive algorithm for finding an optimal value of the feedback gain.

2.2.1 Analytical results

The noise-free ($\varepsilon = 0$) system (8) is stable if both its eigenvalues lie inside of the unit circle in the complex plane. This takes place if $\lambda > -3$ and the coupling strength is in interval

$$-(\lambda + 1)/2 < k < 1. \tag{9}$$

In what follows we suppose that these conditions are satisfied. Then in the presence of noise the system will fluctuate around the stabilized fixed point. The quality of the stabilization can be characterized by the stationary value of the mean square of time-delay difference

$$\sigma^2 = \langle (x_n - x_{n-1})^2 \rangle. \tag{10}$$

Here this variance can be obtained in an explicit analytical form. The variance (10) can be expressed through the correlators $K_0 = \langle x_n^2 \rangle$ and $K_1 = \langle x_n x_{n-1} \rangle$ as follows:

$$\sigma^2 = 2(K_0 - K_1). \tag{11}$$

Supplementing the correlators K_0 and K_1 by additional correlator $K_2 = \langle x_n x_{n-2} \rangle$ and using equations (6) and (7), we obtain a closed system of linear equations:

$$\begin{pmatrix} 1 & -(\lambda+k) & k \\ \lambda+k & -k-1 & 0 \\ k & \lambda+k & -1 \end{pmatrix} \begin{pmatrix} K_0 \\ K_1 \\ K_2 \end{pmatrix} = \begin{pmatrix} \varepsilon^2 \\ 0 \\ 0 \end{pmatrix}. \tag{12}$$

Substituting the solution of this system in equation (11), we get an analytical expression for the variance:

$$\sigma^2(k) = \frac{2\varepsilon^2}{(1-k)(1+2k+\lambda)}. \tag{13}$$

The dependence of the variance (13) on the coupling strength k is shown in Figure 1a. As well as in the previous example, the variance is singular at the boundaries of stability $k = k_{\min} = -(\lambda+1)/2$ and $k = k_{\max} = 1$. However, unlike the previous example, the variance has a minimum at a finite value of $k = k_{\mathrm{op}}$,

$$k_{\mathrm{op}} = (1-\lambda)/4, \tag{14}$$

which represents an optimal value of the coupling strength in the presence of noise. Note that the optimal feedback gain (14) in the presence of noise does not coincide with the value $k_0 = 2 - \lambda - 2\sqrt{1-\lambda}$, which characterizes an optimal feedback gain in the absence of noise. The value k_0 defines the feedback gain at which the absolute value of the leading Floquet multiplier of noise-free system is minimal, i.e. the system approaches the stabilized fixed point in the shortest time.

In what follows we are going to construct an adaptive algorithm for finding the optimal value of the coupling strength k_{op}. Note that the dependence $\sigma^2 = \sigma^2(k)$ is rather complicated: it has two singular points and a flat minimum, which transforms to a plateau in the whole stability interval when $\varepsilon \to 0$ (the dashed line in Fig. 1a). When searching for the optimal coupling strength it is more convenient to deal with the inverse function. Specifically, we introduce the function

$$L(k) = -\frac{\varepsilon^2}{2\sigma^2(k)} \tag{15}$$

in analogy with equation (5), which defines the Lyapunov exponent of a noisy fixed point in the simple model (1). We note that the function $L(k)$ has the minimum at the same k as the function $\sigma^2(k)$. Therefore, we can look for the minimum of the function $L(k)$ instead of the function $\sigma^2(k)$ when searching for the optimal coupling strength. The advantage of the function $L(k)$ is that it has a simple parabolic form $L(k) = -(1-k)(1+2k+\lambda)/4$ (see Fig. 1b), which is well suited for the least-squares polynomial fitting (LSPF) that we apply in our adaptive strategy in the next paragraph.

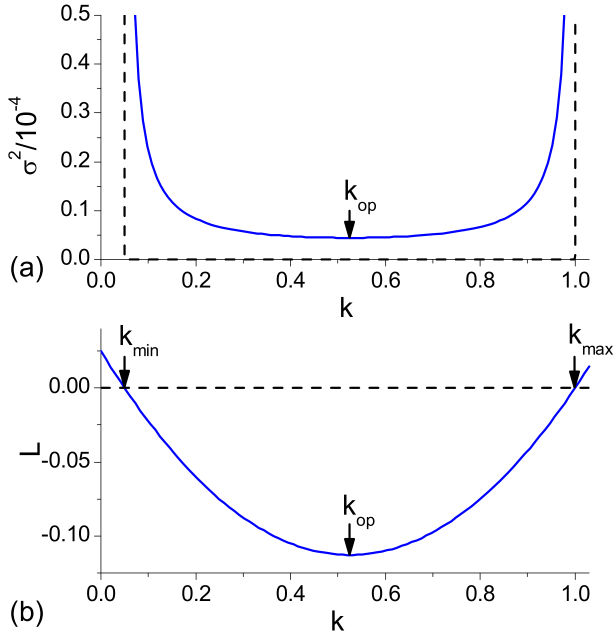

Fig. 1. (a) The dependence of the variance σ^2 on the coupling strength k defined by equation (13) for $\varepsilon = 0.01$ (blue solid curve) and in the limit $\varepsilon \to 0$ (black dashed curve). (b) The dependence L vs. k defined by equations (15) and (13). In both diagrams $\lambda = -1.1$.

2.2.2 Search for the optimal feedback gain using trial points

Now we imagine an experimental situation in which the system equations are unknown. Our aim is to construct an adaptive algorithm, which allows us to determine the optimal value of the coupling strength on the basis of the on-line measured output signal x_n and maintain the optimal coupling even in the presence of a slow variation of the system parameters. We assume that the drift of parameters occurs on such a slow time scale that we can perform on-line statistical analysis of the output signal in an adiabatic approximation, i.e., treating the values of the parameters constant.

Our algorithm is as follows. First we choose some segment $[k_1, k_M]$ of the feedback gain lying in the stability interval $[k_{\min}, k_{\max}]$ and fix in this segment M homogeneously distributed trial points $(k_1, k_2, \ldots, k_M)$, with $k_{j+1} - k_j = \Delta k$, $j = 1, \ldots, M-1$. For each trial value k_j we observe the dynamics of x_n in some time interval N and estimate the variance σ^2 using the time average

$$\sigma^2 = \frac{1}{N} \sum_{n'=n-N}^{n} (x_{n'} - x_{n'-1})^2. \tag{16}$$

Figure 2a shows the dependence of σ^2 on the length of the averaging interval N for $k = 0.5$ and $\lambda = -1.1$. For sufficiently large N, expression (16) provides a convergence to a true theoretical value σ^2 defined by equation (13), which in Figure 2a is presented by a dashed line.

Obtaining $\sigma^2(k_j)$ for a given trial value k_j we use equation (15) to compute the value $L(k_j) = -\varepsilon^2/2\sigma^2(k_j)$. Repeating this procedure for all trial points k_j, we obtain

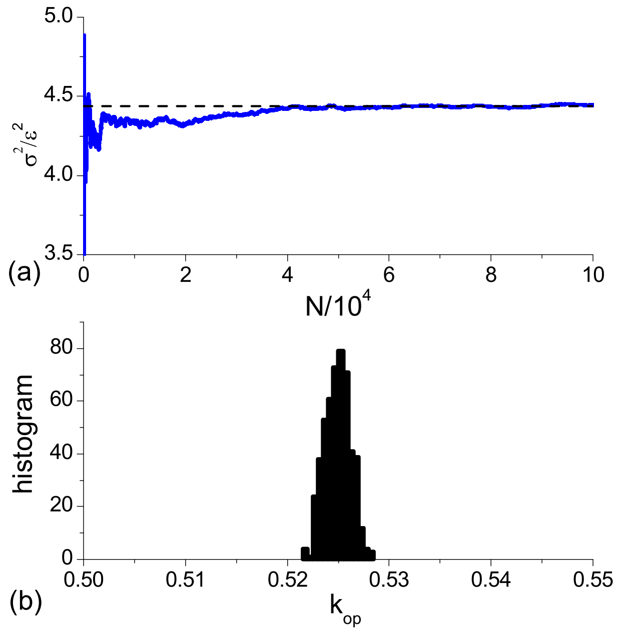

Fig. 2. (a) The convergence of a time average (16) (blue solid curve) to a true value of the variance $\sigma^2(k)$ defined by equation (13) (black dashed line) for $k = 0.5$ and $\lambda = -1.1$. (b) Histogram of optimal values k_{op} obtained from 500 realizations of the adaptive model independent algorithm with $N = 10^4$, $M = 3$ and $\Delta k = 0.175$.

the sequence of M values $L_j = L(k_j)$, $j = 1, \ldots, M$. Having the values (k_j, L_j) we approximate the dependence $L = L(k)$ by the LSPF. Note that the number of trial points M has to be grater than the order of the approximating polynomial. Here we use the second order polynomial, i.e. fit the obtained points (k_j, L_j) by the parabola $L = Ak^2 + Bk + C$. Then the LSPF method yields the coefficients (A, B, C) and the optimal coupling strength is determined from the minimum of parabola as follows:

$$k_{\text{op}} = -B/2A. \tag{17}$$

In the next step, we construct a new sequence of feedback gains $(k_1, k_2, \ldots, k_M)$ centered about the obtained optimal value k_{op} such that $(k_1 + k_M)/2 = k_{\text{op}}$ and $\Delta k = (k_M - k_1)/(M-1)$. Then we repeat the above procedure to obtain the new value of k_{op}. In Figure 2b, we show the histogram of k_{op} values obtained from 500 realizations of this second step of the algorithm with only $M = 3$ trial values of the feedback gain $(k_1, k_2, k_3) = (k_{\text{op}} - \Delta k, k_{\text{op}}, k_{\text{op}} + \Delta k)$ with $\Delta k = 0.175$. We see that all realizations fall very close to the exact optimal value $k_{\text{op}} = 0.525$.

Thus the described algorithm enables us to obtain the optimal value of the feedback gain of the TDFC noisy system without recourse to the system model. The method provides reliable results by using only three trial values of the feedback gain. In the presence of a slow variation of system parameters, an adaptation of the feedback gain to the optimal value can be achieved by repeating the second step of the algorithm through some appropriate time intervals.

2.3 Time-delayed feedback control of a noisy Rössler system

Now we show that the above algorithm is applicable to more complex systems. Consider the Rössler [23] system controlled by time-delayed feedback in the presence of noise:

$$\dot{x} = -y(t) - z(t) + \xi_1(t), \tag{18a}$$
$$\dot{y} = x(t) + ay(t) - k[y(t) - y(t-\tau)] + \xi_2(t), \tag{18b}$$
$$\dot{z} = b + z(t)[x(t) - c] + \xi_3(t). \tag{18c}$$

Here $a = b = 0.2$ and $c = 5.7$ are the system parameters, $k[y(t) - y(t-\tau)]$ is the time-delayed feedback perturbation, and k is the feedback gain. The last terms ξ_1, ξ_2 and ξ_3 in these equations represent the Gaussian white noise perturbations satisfying

$$\langle \xi_i(t) \rangle = 0, \quad \langle \xi_i(t)\xi_j(t') \rangle = \varepsilon^2 \delta_{ij} \delta(t - t'), \tag{19}$$

where the indexes $i, j = 1, 2, 3$. The system parameters are chosen such as to get a chaotic behavior in the absence of noise and feedback perturbation ($\varepsilon = 0$, $k = 0$). We will focus on stabilization of the period-1 UPO with the period $T_1 = 5.88105$ and choose the delay time $\tau = T_1$. We will measure the quality of the stabilization by the time-averaged square of the time-delay difference of the output signal

$$\sigma^2 = \frac{1}{T} \int_{t-T}^{t} [y(t') - y(t' - \tau)]^2 dt', \tag{20}$$

where T is an averaging interval. Unlike the previous example, here we do not have an analytical expression for the dependence $\sigma^2 = \sigma^2(k)$ and the optimal value of the feedback gain that minimizes $\sigma^2(k)$ can be obtained only numerically. As well as in the previous example, we will search for the minimum of the inverse function $L(k)$ defined by equation (15) rather than the function $\sigma^2(k)$.

In Figure 3a, the black dots show the values of the function $L(k)$ obtained by the computation of the variance (20) with $T = 10^5\tau$ for 100 different values of k taken in the stability interval $k_{\min} < k < k_{\max}$, where $k_{\min} \approx 0.124$ and $k_{\max} \approx 0.66$. The solid curve represents the forth-order polynomial fitting. The optimal feedback gain estimated from the minimum of this polynomial is $k_{\text{op}} \approx 0.433$. We interpret this value as a "true" optimal value.

To estimate k_{op} in an experiment with varying parameters, we need an algorithm requiring as less as possible statistical data, which should achieve the goal by using only few trial points and as short as possible averaging time T. For this purpose the suitable algorithm is that described in Section 2.2.2. The results of application of this algorithm for the Rössler system are presented in Figure 3b. Here, we show the histogram of optimal values k_{op} obtained from 500 realizations of the second step of the algorithm with $T = 2000\tau$ and $M = 5$ trial values $(k_1, k_2, k_3, k_4, k_5) = (k_{\text{op}} - 2\Delta k, k_{\text{op}} - \Delta k, k_{\text{op}}, k_{\text{op}} + \Delta k, k_{\text{op}} + 2\Delta k)$, $\Delta k = 0.08$ using the LSPF with the second order polynomial. We see that the histogram is located in a narrow interval close to the true optimal value.

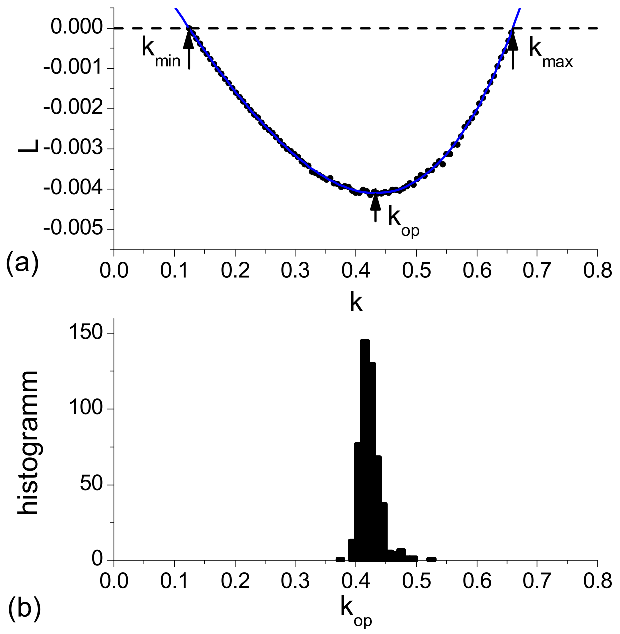

Fig. 3. (a) The dependence of L vs. k for controlled Rössler system (18). The dots denote the values of $L(k)$ numerically obtained by computation of the variance (20) with $T = 10^5\tau$. The solid curve represents the forth order polynomial fitting. The optimal value of the feedback gain obtained from this polynomial is $k_{\text{op}} \approx 0.433$ and boundaries of stability are $k_{\min} \approx 0.124$ and $k_{\max} \approx 0.66$. (b) Histogram of optimal values k_{op} obtained from 500 realizations of the model independent algorithm with $T = 2000\tau$, $M = 5$ and $\Delta k = 0.08$.

We may reduce the time intervals of averaging. In that case, the algorithm would follow a faster drift of parameters but obtained optimal values of the feedback gain would fluctuate around the exact value with larger amplitude, and the corresponding histogram would become broader. Thus this algorithm can be effectively used for noisy continuous-time chaotic systems governed by unknown dynamical laws.

3 Gradient-descent algorithm

The main advantage of the algorithm presented in the previous section is that it does not require the knowledge of the system equations. However, the drawback of the algorithm is a necessity to switch the feedback gain to trial values, which are away from the optimal value. In this section, we consider a gradient-descent algorithm which does not require any trial switches of the feedback gain. Here any initial value of the feedback gain is continuously adjusted towards the optimal value without any trials. However, this algorithm requires the knowledge of the system equations. First, we present the description of this algorithm for a general class of time-continuous systems controlled by time-delayed feedback in the presence of noise and then demonstrate its application for the Rössler system and the normal form of the subcritical Hopf bifurcation.

3.1 Description of the algorithm

We consider a general problem of time-delay feedback controlled system in the presence of noise:

$$\dot{\mathbf{x}}(t) = \mathbf{f}[\mathbf{x}(t)] - k\mathbf{K}[\mathbf{x}(t) - \mathbf{x}(t-\tau)] + \boldsymbol{\xi}(t). \quad (21)$$

Here $\mathbf{x}$ is the state vector of the system, $\mathbf{f}(\mathbf{x})$ describes the vector field of the free system, $\mathbf{K}$ is a control matrix and k is a scalar feedback gain. The last term (vector $\boldsymbol{\xi}$) in equation (21) defines the Gaussian noise, whose components satisfy:

$$\langle \xi_i(t) \rangle = 0, \quad \langle \xi_i(t)\xi_j(t') \rangle = \varepsilon^2 \delta_{ij}\delta(t-t'). \quad (22)$$

We suppose that the free system has an UPO with the period τ. Our aim is to construct an adaptive algorithm that provides an automatic convergence of the feedback gain k to the optimal value k_{op}. By k_{op} we mean the value of the feedback gain corresponding to the minimum of the mean squared control signal

$$\sigma^2 = \frac{1}{T}\int_{t-T}^{t} [\mathbf{x}(t') - \mathbf{x}(t'-\tau)]^2 dt' \quad (23)$$

with sufficiently long averaging time T. The adaptive algorithm should work even in presence of slow variation of the system parameters. We suppose that the equations of motions are known and intend to adapt the feedback gain continuously. Thus the feedback gain k in our algorithm becomes a state dependent variable and we seek to construct an appropriate dynamic equation for this variable.

In order to simplify online computations we follow the ideas of reference [17], and instead of (23) we introduce a mean squared exponentially weighted control signal

$$s = \nu \int_{-\infty}^{t} e^{-\nu(t-t')}[\mathbf{x}(t') - \mathbf{x}(t'-\tau)]^2 dt'. \quad (24)$$

Here ν^{-1} defines the width of the window $[t-\nu^{-1}, t]$ where the running average of the control signal is performed. The parameter ν should satisfy the inequalities $\tau_s \ll \nu^{-1} < \tau_p$ where τ_s is the time scale on which the system dynamics evolves, and τ_p is the time scale on which the system parameters evolve.

Hence we can program k so as to seek the minimum of s. However, the function $s(k)$ as well as $\sigma^2(k)$ has very flat minimum and singularities on the boundaries of the control. For searching the optimal value of the feedback gain, more appropriate is the inverse function (15), which can be expressed as $L(k) = -\varepsilon^2/2s(k)$. Since the derivative dL/dk is negative for $k < k_{\text{op}}$ and positive for $k > k_{\text{op}}$, the optimal value k_{op} can be attained by the following gradient descent relaxation,

$$\frac{dk}{dt} = -\beta\frac{dL}{dk} \equiv -\beta\varepsilon^2\frac{s_k(t)}{2s^2(t)}, \quad (25)$$

where β is a parameter that determines the relaxation time scale and $L(k)$ may be viewed as a potential function for the gradient flow (25). The function s_k is the derivative of the function s with respect to k: $s_k \equiv \partial s/\partial k$. The differential equation for the function s can be obtained by differentiating equation (24) with respect to time:

$$\dot{s} = \nu[\mathbf{x}(t) - \mathbf{x}(t-\tau)]^2 - \nu s. \quad (26)$$

Accordingly, the differential equation for the function s_k can be derived by differentiating equation (26) with respect to k:

$$\dot{s_k} = 2\nu[\mathbf{x}(t) - \mathbf{x}(t-\tau)] \cdot [\mathbf{x}_k(t) - \mathbf{x}_k(t-\tau)] - \nu s_k, \quad (27)$$

where $\mathbf{x}_k$ is the derivative of the state vector with respect to k: $\mathbf{x}_k \equiv \partial \mathbf{x}/\partial k$. Finally, we obtain the equation for this derivative by differentiating equation (21) with respect to k:

$$\dot{\mathbf{x}}_k(t) = D\mathbf{f}[\mathbf{x}(t)]\mathbf{x}_k(t) - k\mathbf{K}[\mathbf{x}_k(t) - \mathbf{x}_k(t-\tau)] - \mathbf{K}[\mathbf{x}(t) - \mathbf{x}(t-\tau)]. \quad (28)$$

Here $D\mathbf{f}[\mathbf{x}(t)]$ is the Jacobian matrix of the control-free system.

Equations (25)–(28) define completely our adaptive algorithm for the system (21).

Note some similarities and differences of this algorithm with the speed-gradient method presented in reference [21]. The main difference between these algorithms is that the speed-gradient method minimizes the current square of the control signal, while our algorithm minimizes the mean square of the control signal. The speed-gradient method is easier to implement and it can shift the feedback gain from an unstable region into the stable one, but it cannot provide the convergence of the feedback gain to the optimal value in the presence of noise. Our algorithm works only for initial values of the feedback gain lying in the stability interval, however, it provides the convergence of the feedback gain to the optimal value. In real experiments, both algorithms can be used together, the speed gradient method – for finding the stable range, and the gradient descent method – for minimizing the mean squared difference between the current and delayed state of the system. Thus both algorithms can be considered as complimentary to each other.

3.2 Application for the Rössler system

Now we demonstrate the efficiency of the gradient descent algorithm for the noisy Rössler system (18). The control matrix in equations (18) represent the diagonal matrix $\mathbf{K} = \mathrm{diag}(010)$, which means that only $y(t)$ variable of the Rössler system (18) is employed in the TDFC force. Here, we will measure the quality of control by the mean squared difference $\langle [y(t) - y(t-\tau)]^2 \rangle$. Then the gradient descent algorithm for the Rössler system (18) can be presented by

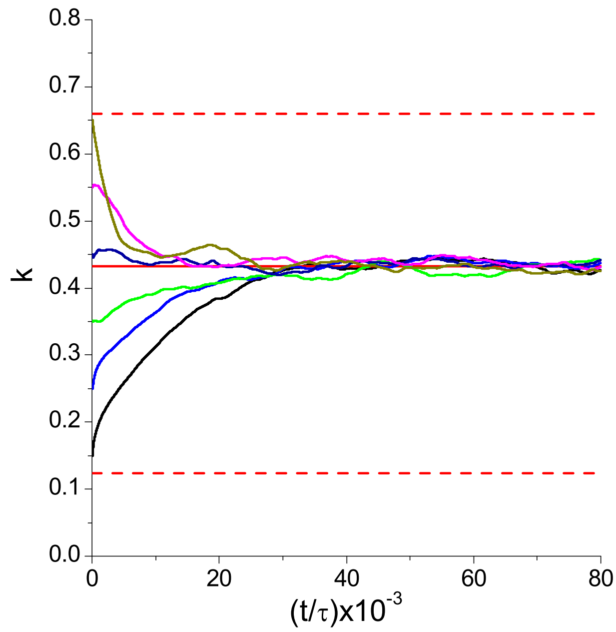

Fig. 4. Convergence of the feedback gain to the optimal value for a noisy Rössler system. The results are obtained by numerical integration of equations (18) and (29) for different initial values of the feedback gain. The initial values of the variables s, s_k, x_k, y_k and z_k are zeros. The values of the parameters are: $\tau = 5.88105$, $\varepsilon = 0.001$, $\beta = 0.0005$ and $\nu = 1/1000\tau$. The horizontal solid line shows the optimal value of the feedback gain $k_{\mathrm{op}} \approx 0.433$. The dashed horizontal lines indicate the boundaries of the control $k_{\min} \approx 0.124$ and $k_{\max} \approx 0.66$.

the following set of differential equations:

$$\dot{k}(t) = -\beta\varepsilon^2 s_k(t)/2s^2(t), \quad (29a)$$

$$\dot{s}(t) = \nu[y(t) - y(t-\tau)]^2 - \nu s(t), \quad (29b)$$

$$\dot{s_k}(t) = 2\nu[y(t) - y(t-\tau)][y_k(t) - y_k(t-\tau)] - \nu s_k(t), \quad (29c)$$

$$\dot{x}_k(t) = -y_k(t) - z_k(t), \quad (29d)$$

$$\dot{y}_k(t) = x_k(t) + a y_k(t) - k[y_k(t) - y_k(t-\tau)] - [y(t) - y(t-\tau)], \quad (29e)$$

$$\dot{z}_k(t) = z_k(t)[x(t) - c] + z(t)x_k(t). \quad (29f)$$

The results of integration of equations (18) and (29) for different initial values of the feedback gain are presented in Figure 4. We see that the feedback gain starting from any initial value in the domain of stability approaches the optimal value $k_{\mathrm{op}} \approx 0.433$ and fluctuates in some finite interval about this value.

Note that our algorithm is insensitive to initial conditions of the system's variables as well as control variables $s(0)$ and $s_k(0)$, however it requires that the initial value of the feedback gain has to be inside the interval of the successful control.

3.3 Application for the normal form of the subcritical Hopf bifurcation

To demonstrate the universality of the proposed gradient descent algorithm, we consider its application to another system, the normal form of the subcritical Hopf bifurcation, which has been recently presented as an example of autonomous system that refutes the odd number limitation of the TDFC [4]. In terms of the complex-valued variable $z(t)$ the equation of motion with time-delayed feedback control and noise reads [4]:

$$\dot{z}(t) = (\lambda + i)z(t) + (1 + i\gamma)|z(t)|^2 z(t) - k\mathrm{e}^{i\alpha}[z(t) - z(t-\tau)] + \xi(t), \quad (30)$$

where λ and γ are real parameters. The parameter λ determines the distance from the Hopf bifurcation point of the uncontrolled system, and the parameter γ governs the dependence of the oscillation frequency on the amplitude. The Hopf frequency is normalized to unity. Control is governed by the real-valued feedback gain k and a phase α, which is a crucial quantity that allows us to overcome the odd number limitation [4]. Finally, $\xi(t) = \xi_1(t) + i\xi_2(t)$ describes the white Gaussian noise that satisfies $\langle \xi_i(t) \rangle = 0$ and $\langle \xi_i(t)\xi_j(t') \rangle = \varepsilon^2 \delta_{i,j}\delta(t-t')$ for $i, j = 1, 2$.

For $\lambda < 0$ the system without control and noise ($k = 0$, $\varepsilon = 0$) has an unstable periodic orbit $z(t) = R\exp(i\Omega t)$ with amplitude $R = \sqrt{-\lambda}$ and frequency $\Omega = (1 - \gamma\lambda)$. In order to stabilize this orbit we choose the delay time of the TDFC equal to the period of the orbit: $\tau = 2\pi/\Omega$. Our aim is to demonstrate the adaptive tuning of the feedback gain to the optimal value via the gradient descent algorithm. The equations of this algorithm for the system (30) read:

$$\dot{k}(t) = -\beta\varepsilon^2 s_k(t)/2s^2(t), \quad (31a)$$

$$\dot{s}(t) = \nu|z(t) - z(t-\tau)|^2 - \nu s(t), \quad (31b)$$

$$\dot{s_k}(t) = 2\nu[(x(t)) - x(t-\tau)][x_k(t) - x_k(t-\tau)] + 2\nu[(y(t)) - y(t-\tau)][y_k(t) - y_k(t-\tau)] - \nu s_k(t), \quad (31c)$$

$$\dot{z}_k(t) = (\lambda + i)z_k(t) + (1 + i\gamma)|z(t)|^2 z_k(t) + 2(1 + i\gamma)[x(t)x_k(t) + y(t)y_k(t)]z(t) - ke^{i\alpha}[z_k(t) - z_k(t-\tau)] - e^{i\alpha}[z(t) - z(t-\tau)], \quad (31d)$$

where $x = \mathrm{Re}(z)$, $y = \mathrm{Im}(z)$, $x_k = \mathrm{Re}(z_k)$ and $y_k = \mathrm{Im}(z_k)$. The results of integration of equations (30) and (31) for different initial values of the feedback gain are presented in Figure 5. Again, the feedback gain starting from any initial value in the domain of stability approaches the optimal value k_{op} and fluctuates in rather narrow interval about this value.

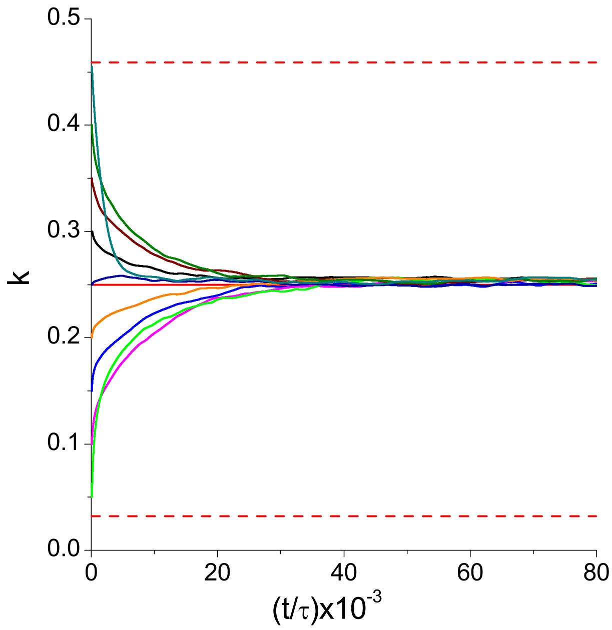

Fig. 5. Noisy normal form of the subcritical Hopf bifurcation: convergence of the feedback gain to the optimal value. The results are obtained by numerical integration of equations (30) and (31) for different initial values of the feedback gain. The parameters are: $\lambda = -0.005$, $\gamma = -10.0$, $\varepsilon = 10^{-4}$, $\beta = 0.0001$ and $\nu = 1/1000\tau$. The horizontal solid line shows the optimal value of the feedback gain $k_{\mathrm{op}} \approx 0.25$. The dashed horizontal lines indicate the boundaries of the control $k_{\mathrm{min}} \approx 0.032$ and $k_{\mathrm{max}} \approx 0.46$.

4 Conclusions

We have proposed two adaptive algorithms for finding the optimal value of the feedback gain in noisy systems controlled by time-delayed feedback method. In the presence of noise the controlled system does not settle exactly on the stabilized periodic orbit but fluctuates in its vicinity. The optimal feedback gain minimizes the fluctuations caused by noise. In both algorithms, the optimal value of the feedback gain is determined from the minimum of the mean squared difference between the actual state of the system and the state of the system delayed by the period of the controlled unstable periodic orbit. The algorithms have different advantages and shortcomings so that their choice depends on the specific experimental conditions. If the system equations are unknown then the first algorithm based on trial switches of the feedback gain and the least-squares polynomial fitting is preferable. If the system equations are known, the second algorithm that uses the gradient descent method is more appropriate. This algorithm does not require any trial switches of the feedback gain; here any initial value of the feedback gain is continuously adjusted towards the optimal value.

The efficacy of the algorithms is demonstrated for different specific systems, namely, a simple linear map, the Rössler system and the normal form of the subcritical Hopf bifurcation. The periodic orbit of the latter system satisfies the criterion of the odd number limitation. This means that our algorithms are applicable not only for typical orbits available for the time-delayed feedback control, but also for periodic orbits which are beyond the odd number limitation.

In this paper, we assumed that the period of controlled periodic orbit is known. However, the gradient descent algorithm can be simply generalized for the case of unknown period as well by using the ideas of reference [17].

This research was funded by the European Social Fund under the Global Grant measure (Grant No. VP1-3.1-ŠMM-07-K-01-025).

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