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Control of chaos via extended delay feedback

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Abstract

We present a linear analysis for a recently proposed modification of the delay feedback control technique that allows one to stabilize unstable periodic orbits of a strange attractor over a large domain of parameters. The method uses a continuous feedback loop incorporating information from many previous states of the system in a form closely related to the amplitude of light reflected from a Fabry–Perot interferometer. We illustrate the possibility of stabilizing high-periodic orbits and fixed points with large values of Lyapunov exponents.

An interesting and challenging research subject recently arisen in the field of nonlinear dynamical systems is the control of chaos, namely, the investigation of bringing order into chaos. Roughly speaking there are two kinds of ways to control chaos: feedback control [1–22] and nonfeedback control [23–27]. Nonfeedback control changes the controlled orbit of the system and requires comparatively large perturbations. In this Letter we focus on feedback control. For chaotic systems, such a type of control has been used for the first time by Ott, Grebogy and Yorke (OGY) [1]. The key idea is to take advantage of unstable periodic orbits (UPOs) embedded in a strange attractor. As the system approaches an UPO, the strength of the perturbation required to keep it there vanishes, so that the smallness of the feedback signal is limited only by the noise level in the system. The OGY method does not require any a priori analytical knowledge of the system dynamics and has been successfully applied to various physical experiments, including magnetic ribbon [2], spin wave [3], chemical [4], electric diode [5], laser [6], cardiac [7] and neuronal [8] systems. The OGY method and its various modifications [4–12] (see also Ref. [13] for a survey) are discrete in time since they deal with the Poincaré map of the system. Because of that they are sensitive to noise [1]. None of these techniques can be scaled up to significantly higher frequencies since they involve discontinuous adjustment of the control parameter.

An alternative approach based on continuous-time control has been suggested by the present author [14–16]. The method deals with a chaotic system that can be simulated by a set of ordinary nonlinear differential equations [14],

$$\dot{y} = P(y, x) + F(t), \quad \dot{x} = Q(y, x). \tag{1}$$

We imagine that Eqs. (1) are unknown, but some scalar variable $y(t)$ can be measured as a system output. The vector $x(t)$ describes the remaining variables of the system that are not available or are not of interest for observation. The complete state of the unperturbed

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system is determined by the vector $\{y(t), x(t)\}$ in an n_Γ-dimensional phase space Γ. $F(t)$ denotes an external continuous-time perturbation. The idea behind it is to construct this perturbation in such a way that it does not change the desired UPO of the system, but only changes the corresponding Lyapunov exponents so that the orbit becomes stable. Different types of perturbations satisfying this requirement have been considered [14,16,17]. The most interesting from an experimental point of view is the technique based on the delay feedback perturbation [14],

$$F(t) \equiv F(y(t), y(t-\tau)) = K[y(t-\tau) - y(t)]. \tag{2}$$

This perturbation vanishes on the period-k UPO if the delay time τ coincides with the period T_k of this UPO, $\tau = T_k$. Thus, if stabilization is successful there is no power dissipated in the feedback loop. The method does not require any preliminary or on-line analysis of the system dynamics and can be implemented in any experiment by a purely analog technique. The method has been successfully applied to nonautonomous [18,19] as well as autonomous [20] electronic chaos oscillators and to a laser system [21]. Unfortunately, the domain of the system parameters over which control can be achieved is limited [14]. The method fails for high-period orbits.

To overcome this problem, Socolar, Sukow and Gauthier [22] have recently proposed a generalization of the feedback law (2) utilising the information from many previous states of the system,

$$F(t) = K\left((1-R)\sum_{m=1}^{\infty} R^{m-1} y(t-m\tau) - y(t)\right), \tag{3}$$

where $0 \leqslant R < 1$ and K are experimentally adjustable constants. For any R, perturbation (3) vanishes when the system is on the UPO since $y(t-m\tau) = y(t)$ for all m if $\tau = T_k$. Thus, this feedback also ensures small values of the perturbation in the case of successful control. It is interesting to note [22] that Eq. (3) represents precisely the signal reflected from an interferometer consisting of mirrors with reflectivity R, spaced in such a way that the round-trip transit time in the cavity is equal to the period of the UPO [28]. Hopefully, a Fabry–Perot interferometer can be used to control chaos in optical systems. At $R = 0$ Eq. (3) turns into the original feedback law (2). It has been suggested [22] that the methods based on the original feedback law (2) and on the modified feedback (3) should be called time delay autosynchronization (TDAS) and extended TDAS (ETDAS), respectively. We do so in this paper.

The authors of Ref. [22] have demonstrated the advantages of ETDAS experimentally by applying it to a high-frequency chaotic electrical circuit: a diode resonator [29] driven at 10 MHz. It turns out that experimental implementation of ETDAS is very easy. The infinite series in Eq. (3) was generated with *a single* delay line. They managed to stabilize UPOs over a wide range of system parameters far away beyond the threshold of chaotic instability, where the original TDAS scheme fails. The aim of this Letter is to amplify these investigations by a theoretical analysis based on the calculation of Lyapunov exponents for various dynamical systems under ETDAS control. This linear characteristic determines the rate at which the system approaches or diverges from the UPO and serves as a good criterion of control. It defines not only the domain of the system parameters where control is possible but also the quality of the control in this domain.

Before defining the variational equations for Lyapunov exponents, let us rewrite Eq. (3) in a more convenient form,

$$F(t) = K[(1-R)S(t-\tau) - y(t)],$$
$$S(t) = y(t) + RS(t-\tau). \tag{4}$$

Here we have replaced the sum $S(t) \equiv \sum_{m=0}^{\infty} R^m y(t-m\tau)$ by an equivalent delay equation. If we are interested in solutions of this equation at times $t \geqslant 0$, it becomes necessary to define the initial sum $S(t)$ in the entire interval $[-\tau, 0]$, $S(\Theta) = S_{\rm in}(\Theta)$, $-\tau \leqslant \Theta \leqslant 0$, where $S_{\rm in}(\Theta)$ is a given continuous initial function in a suitable function space $\mathcal{C}$. The state of the delay system at time t can be described by an extended state vector $S_t \in \mathcal{C}$ constructed in the interval $[t-\tau, t]$ according to the prescription $S_t(\Theta) = S(t+\Theta)$, $-\tau \leqslant \Theta \leqslant 0$ [30]. To ensure the uniqueness of the solutions of the perturbed system (1) one requires to extend the initial phase space Γ by the infinite-dimensional function space $\mathcal{C}$.

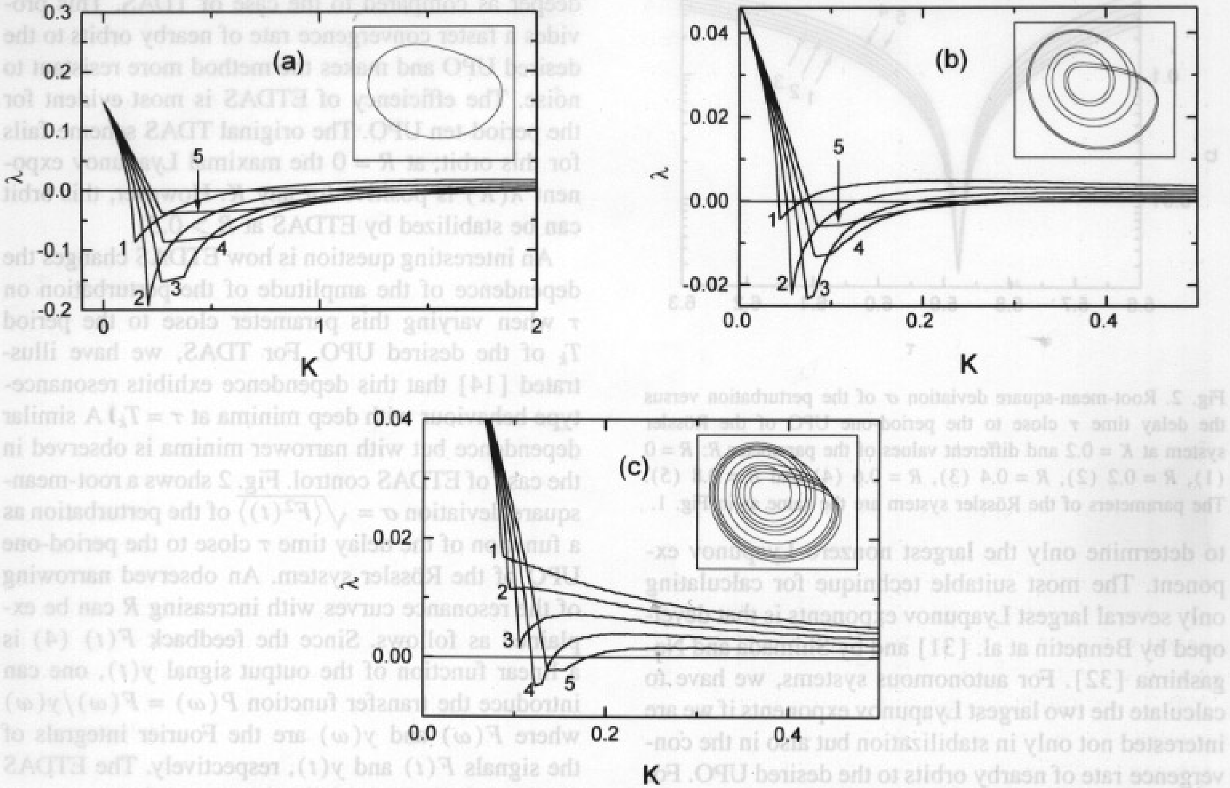

Fig. 1. Maximal nonzero Lyapunov exponent λ of the Rössler system (6) versus the weight K of the perturbation for (a) period-one UPO ($\tau = T_1 = 5.88$), (b) period-six UPO ($\tau = T_6 = 35.01$) and (c) period-ten UPO ($\tau = T_{10} = 58.48$) at various values of the parameter R: $R = 0$ (1), $R = 0.2$ (2), $R = 0.4$ (3), $R = 0.6$ (4) and $R = 0.8$ (5). The inserts show the x–y phase portraits of the stabilized orbits. The parameters of the Rössler system (Eq. (6)) are $a = 0.2$, $b = 0.2$, $c = 5.7$.

The Lyapunov exponents of Eqs. (1), (4) corresponding to the period-k UPO $\{y_k(t), x_k(t)\} = \{y_k(t+T_k), x_k(t+T_k)\}$ are determined by variational equations,

$$\delta\dot{y} = \delta y \frac{\partial}{\partial y} P(y_k, x_k) + \delta x \frac{\partial}{\partial x} P(y_k, x_k) + K[(1-R)\delta S(t-T_k) - \delta y(t)],$$

$$\delta\dot{x} = \delta y \frac{\partial}{\partial y} Q(y_k, x_k) + \delta x \frac{\partial}{\partial x} Q(y_k, x_k),$$

$$\delta S(t) = \delta y(t) + R\delta S(t-T_k). \quad (5)$$

Here $\delta y = y - y_k$, $\delta x = x - x_k$, and $\delta S = S - S_k$ define the deviation of the system from a periodic orbit. Because of the infinite dimension of the phase space, the system has an infinite number of Lyapunov exponents. In the numerical integration of Eqs. (5) or (1), (4), we are able to consider only a discrete, finite-dimensional version of these equations. Then the dimension n_C of the functional space C becomes finite, $n_C = \tau/h$, where h is the time step of the integration [2]. The complete state of the system is defined by the $(n_\Gamma + n_C)$-dimensional vector $\{y(t), x(t), S_0(t), S_1(t), \ldots, S_{n_C-1}(t)\}$ where $S_i(t) = S(t - ih\tau)$, $i = 0, 1, \ldots, n_C - 1$. It is convenient to perform the numerical integration by the Runge–Kutta method of second order. Unlike the methods of higher order, it does not require the knowledge of the delayed signal $S(t-\tau)$ at the moments inside the integration intervals.

For our purpose, we do not need to estimate the whole spectrum of Lyapunov exponents; it suffices

[2] The step h is chosen such that τ/h is an integer number.

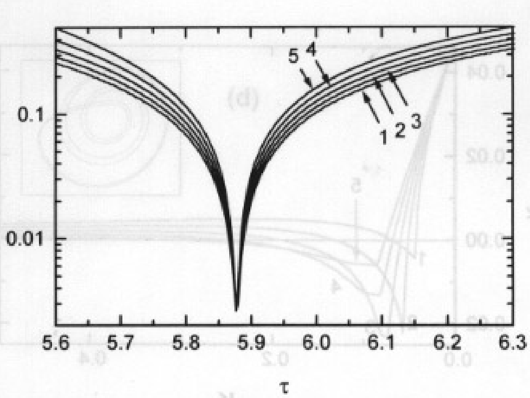

Fig. 2. Root-mean-square deviation σ of the perturbation versus the delay time τ close to the period-one UPO of the Rössler system at $K = 0.2$ and different values of the parameter R: $R = 0$ (1), $R = 0.2$ (2), $R = 0.4$ (3), $R = 0.6$ (4) and $R = 0.8$ (5). The parameters of the Rössler system are the same as in Fig. 1.

to determine only the largest nonzero Lyapunov exponent. The most suitable technique for calculating only several largest Lyapunov exponents is that developed by Bennetin at al. [31] and by Shimada and Nagashima [32]. For autonomous systems, we have to calculate the two largest Lyapunov exponents if we are interested not only in stabilization but also in the convergence rate of nearby orbits to the desired UPO. For such systems, the largest Lyapunov exponent of the stabilized UPO is equal to zero. For nonautonomous systems, it always suffices to determine only the first largest Lyapunov exponent since these systems do not have a zero exponent associated with a tangent to the flow direction.

In Figs. 1–3, we present the results of ETDAS control for two chaotic oscillators. The first model is the autonomous Rössler system [33],

$$\dot{x} = -y - z, \quad \dot{y} = x + ay + F(t),$$
$$\dot{z} = b + z(x - y), \tag{6}$$

where $F(t)$ is defined by Eq. (4). Fig. 1 shows the dependence of the maximal nonzero Lyapunov exponent λ on the weight K of the perturbation for different values of the parameter R. Cases (a)–(c) correspond to period-one, period-six, and period-ten UPOs, respectively. In all cases, ETDAS significantly improves the control in comparison to the original TDAS scheme ($R = 0$). The interval of K where control is achievable ($\lambda(K) < 0$) increases with increasing R. Moreover, if R is not very large, the minimum of $\lambda(K)$ is deeper as compared to the case of TDAS. This provides a faster convergence rate of nearby orbits to the desired UPO and makes the method more resistant to noise. The efficiency of ETDAS is most evident for the period-ten UPO. The original TDAS scheme fails for this orbit; at $R = 0$ the maximal Lyapunov exponent $\lambda(K)$ is positive for any K. However, this orbit can be stabilized by ETDAS at $R > 0.5$.

An interesting question is how ETDAS changes the dependence of the amplitude of the perturbation on τ when varying this parameter close to the period T_k of the desired UPO. For TDAS, we have illustrated [14] that this dependence exhibits resonance-type behaviour with deep minima at $\tau = T_k$. A similar dependence but with narrower minima is observed in the case of ETDAS control. Fig. 2 shows a root-mean-square deviation $\sigma = \sqrt{\langle F^2(t)\rangle}$ of the perturbation as a function of the delay time τ close to the period-one UPO of the Rössler system. An observed narrowing of the resonance curves with increasing R can be explained as follows. Since the feedback $F(t)$ (4) is a linear function of the output signal $y(t)$, one can introduce the transfer function $P(\omega) = F(\omega)/y(\omega)$ where $F(\omega)$ and $y(\omega)$ are the Fourier integrals of the signals $F(t)$ and $y(t)$, respectively. The ETDAS feedback leads to the following expression,

$$P(\omega) = K\frac{e^{i\omega\tau} - 1}{1 - R\,e^{i\omega\tau}}. \tag{7}$$

Since the perturbation vanishes when the system is on the UPO, the frequency components of the output signal at exact multiples of the UPO period are filtered out of the feedback $P(2\pi m/\tau)$, $m = 0, \pm 1, \pm 2, \ldots$ at $\tau = T_k$. Close to these frequencies, $|P(\omega)|$ has resonance minima, the width of which decreases with increasing R. This causes the narrowing of the resonance curves presented in Fig. 2. Note, that a frequency-domain analysis of the controlled system provides a partial answer to the question why ETDAS control is so successful [22]. As R tends to 1, the transfer function $P(\omega)$ becomes a plateau $P(\omega) \approx -K$ almost for all frequencies except for narrow windows close to the points $\omega = \omega_m = 2\pi/T_k$, $m = 0, \pm 1, \pm 2, \ldots$. Thus, as R is increased the feedback becomes more sensitive almost for all frequencies except for those belonging to the UPO.

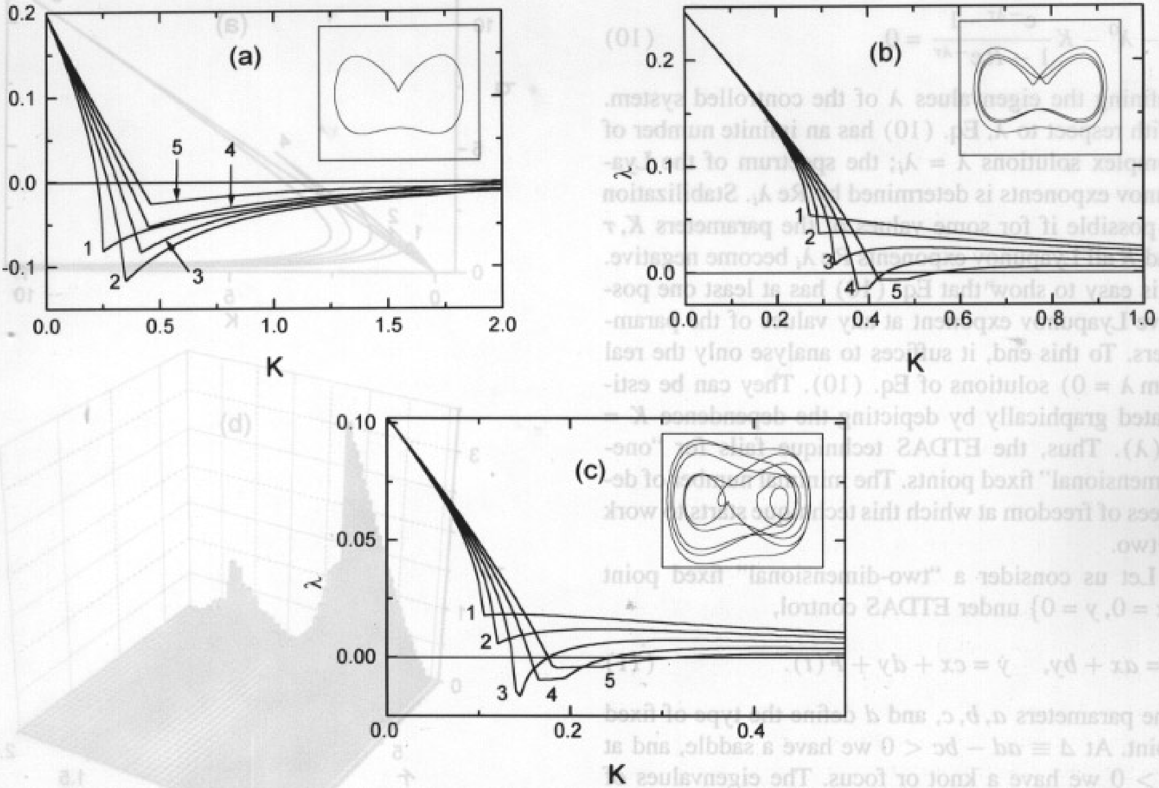

Fig. 3. Maximal Lyapunov exponent λ of the Duffing oscillator (8) versus the weight K of the perturbation for (a) period-one UPO ($\tau = T_1 = 2\pi/\omega$), (b) period-three UPO ($\tau = T_3 = 6\pi/\omega$) and (c) period-five UPO ($\tau = T_5 = 10\pi/\omega$) at various values of the parameter R: $R = 0$ (1), $R = 0.2$ (2), $R = 0.4$ (3), $R = 0.6$ (4) and $R = 0.8$ (5). The parameters of the Duffing system (Eq. (8)) are $a = 2.5$, $\omega = 1$, $d = 0.02$.

The second model that we have analysed is the nonautonomous Duffing oscillator [34],

$$\dot{x} = y,$$

$$\dot{y} = -dy + x - x^3 + a\cos(\omega t) + F(t). \qquad (8)$$

The dependence of the Lyapunov exponents on K for period-one, period-three, and period-five orbits is shown in Fig. 3. Qualitatively the results are similar to those presented in Fig. 1. The increase of the parameter R widens the interval of the parameter K where stabilization is achievable, and makes possible the stabilization of high-period orbits for which the TDAS technique fails.

Some insight into the source of the improved performance obtained with ETDAS can be gained by considering the problem of stabilizing unstable steady states of the system. In Ref. [14], we have illustrated the possibility of stabilizing unstable fixed points by the TDAS technique. Here we consider how ETDAS changes the conditions of this stabilization. Remarkably, this analysis is based on linear equations that can be treated analytically. The advantages of the discrete version of ETDAS when stabilizing the fixed points of discrete maps have been illustrated by the authors of Ref. [22]. Unlike this, our investigation is based on continuous-time systems.

Let us start our analysis from a simple "one-dimensional" unstable fixed point $y = 0$ under ETDAS control,

$$\dot{y} = \lambda^0 y + F(t). \qquad (9)$$

Here $\lambda^0 > 0$ is the eigenvalue of the unperturbed fixed point and $F(t)$ is defined by Eq. (4). Using the substitution $y(t) = C\,e^{\lambda t}$ one obtains the equation

$$\lambda - \lambda^0 - K\frac{e^{-\lambda\tau} - 1}{1 - R\,e^{-\lambda\tau}} = 0, \qquad (10)$$

defining the eigenvalues λ of the controlled system. With respect to λ, Eq. (10) has an infinite number of complex solutions $\lambda = \lambda_i$; the spectrum of the Lyapunov exponents is determined by $\mathrm{Re}\,\lambda_i$. Stabilization is possible if for some values of the parameters K, τ and R all Lyapunov exponents $\mathrm{Re}\,\lambda_i$ become negative. It is easy to show that Eq. (10) has at least one positive Lyapunov exponent at any values of the parameters. To this end, it suffices to analyse only the real ($\mathrm{Im}\,\lambda = 0$) solutions of Eq. (10). They can be estimated graphically by depicting the dependence $K = K(\lambda)$. Thus, the ETDAS technique fails for "one-dimensional" fixed points. The minimal number of degrees of freedom at which this technique starts to work is two.

Let us consider a "two-dimensional" fixed point $\{x = 0, y = 0\}$ under ETDAS control,

$$\dot{x} = ax + by, \quad \dot{y} = cx + dy + F(t). \qquad (11)$$

The parameters a, b, c, and d define the type of fixed point. At $\Delta \equiv ad - bc < 0$ we have a saddle, and at $\Delta > 0$ we have a knot or focus. The eigenvalues of the controlled system satisfy

$$\lambda^2 - (a+d)\lambda + \Delta + K(a - \lambda)\frac{e^{-\lambda\tau} - 1}{1 - R\,e^{-\lambda\tau}} = 0. \qquad (12)$$

An analysis similar to that described above shows that at $\Delta < 0$ and any values of the other parameters Eq. (12) has at least one real positive eigenvalue. It means that the "two-dimensional" saddle cannot be stabilized by the ETDAS technique. However, ETDAS works for any "two-dimensional" focus or knot. Fig. 4 illustrates the domains of successful stabilization of the fixed point at $a = 0$, $b = -1$, $c = 1$, and $d > 0$ being the control parameter. In this case, the eigenvalues of the unperturbed fixed point are $\lambda^0_{1,2} = \frac{1}{2}d \pm \sqrt{\frac{1}{4}d^2 - 1}$. At $0 < d < 2$ we have an unstable focus with the Lyapunov exponent equal to $\frac{1}{2}d$ and eigenfrequency $\omega^0 = \sqrt{1 - \frac{1}{4}d^2}$. At $d > 2$ the fixed point turns into an unstable knot with the maximal Lyapunov exponent equal to $\frac{1}{2}d + \sqrt{\frac{1}{4}d^2 - 1}$. In both cases, the maximal Lyapunov exponent of the fixed point increases with

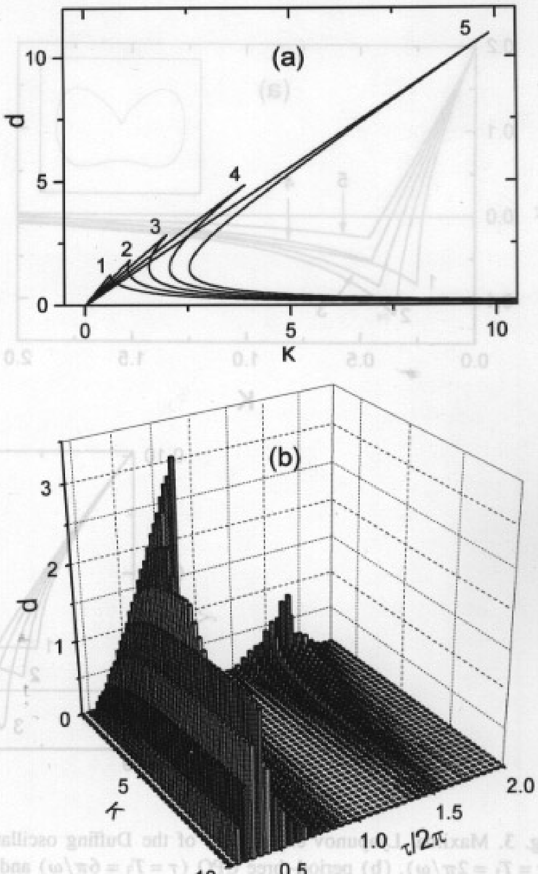

Fig. 4. Domain of stabilization of a "two-dimensional" fixed point (a) in the two-dimensional plane of the parameters, K–d, at a fixed value of the parameter $\tau = \pi$ and various values of the parameter R: $R = 0$ (1), $R = 0.2$ (2), $R = 0.4$ (3), $R = 0.6$ (4), $R = 0.8$ (5) and (b) in the three-dimensional space of the parameters τ–K–d, at a fixed value of the parameter $R = 0.5$. The other parameters of the fixed point (Eq. (11)) are $a = -1$, $b = 0$, $c = 1$.

increasing d. Thus, d characterizes the degree of unstability of the unperturbed fixed point. Fig. 4a shows the regions of effective control in the K–d plane for various values of the parameter R and a fixed value of the parameter τ. The regions correspond to the condition $\mathrm{Re}\,\lambda_i < 0$, where $\lambda = \lambda_i$ denotes all possible solutions of Eq. (12). As can be seen from the figure, the TDAS scheme ($R = 0$) works only for unstable focus with a Lyapunov exponent restricted by $d < 1.216$. The ETDAS extends the operating range of the method for an unstable knot with an arbitrarily large Lyapunov exponent. The greater R, the larger the interval of the parameter d over which the method works. As R tends to 1 this interval becomes infinitely large.

Although ETDAS opens up possibilities for stabilizing a high-unstable fixed point, in this case it requires a careful fitting of the control parameters. For large d, the parameter R has to be chosen close to 1 and the stabilization turns out to be only possible in a narrow interval of the parameter K. A similar effect is observed when stabilizing UPOs of chaotic systems. As is seen from Figs. 1 and 3, the stabilization of high-period UPOs can be only achieved with a large value of the parameter R and only in a narrow interval of the parameter K. Another general feature of ETDAS that holds for the fixed point as well as for the UPOs is that the interval of the parameter K corresponding to stabilization shifts towards larger values of K and increases when increasing R. Thus, Eqs. (11), (3) can be considered as the simplest mathematical model illustrating the general properties of ETDAS control.

In the stabilization of UPOs, the parameter τ has to be fixed equal to the period of the desired UPO. By contrast, in the stabilization of the fixed point, the parameter τ can be chosen arbitrarily. Fig. 4b shows the influence of this parameter on the efficiency of stabilizing the fixed point. The domain of stabilization is illustrated in the three-dimensional parameter space τ, K and d at a fixed value of the parameter R. The vertical bars show the local intervals of the parameter d for which the stabilization of the fixed point is possible and hence define the local efficiency of the method. It is evident from the figure that the efficiency of the method has a resonance-type dependence on τ. The method is most efficient for $\tau = (2m+1)\pi$, $m = 0, 1, 2, \ldots$, and less efficient for $\tau = 2m\pi$, $m = 0, 1, 2, \ldots$. The transfer function (7) of the feedback is again useful in understanding this resonance dependence. This function has maxima at $\omega\tau = (2m+1)\pi$, $m = 0, 1, 2, \ldots$, and minima at $\omega\tau = 2m\pi$, $m = 0, 1, 2, \ldots$. Let us assume, for simplicity, that the fixed point is a focus with a small value of the parameter d. Then the main frequency transmitted into the feedback loop will be the eigenfrequency of the focus $\omega = \omega^0 \approx 1$. Thus, the values of τ corresponding to the maximal and minimal sensitivity of feedback coincide with those corresponding to the maximal and minimal efficiency of the method.

Note, that the problem of stabilizing fixed points by TDAS or ETDAS techniques is, maybe, more important for various applications than the problem of stabilizing UPOs. These techniques do not require any knowledge of the location of the fixed point in phase space, and can work for systems whose parameters vary slowly with time. Here we have restricted ourselves to the analysis of a "two-dimensional" fixed point. A similar analysis can be performed for a fixed point embedded in a high-dimensional phase space.

In conclusion, the linear analysis of a recently proposed chaos control method based on ETDAS technique shows its significant advantages over the original TDAS method. The method allows for stabilization of UPOs with a large value of the Lyapunov exponents and high-period UPOs. It can work in the domain of parameters of chaotic systems far away from the threshold of chaotic instability, where the original TDAS technique fails. The universality of the results is demonstrated with autonomous and nonautonomous chaotic systems, and with a simple model describing the problem of stabilizing an unstable fixed point.

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